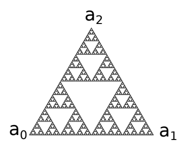
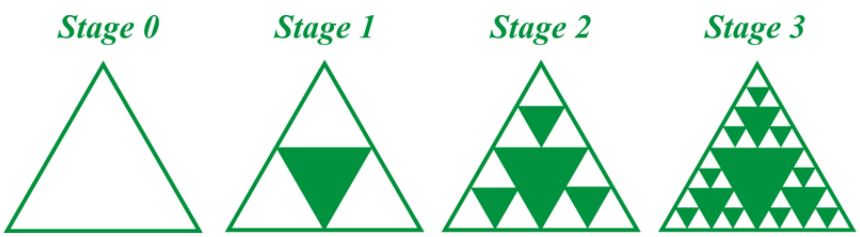
Lecture Notes: 08.01.2013

January 14, 2013

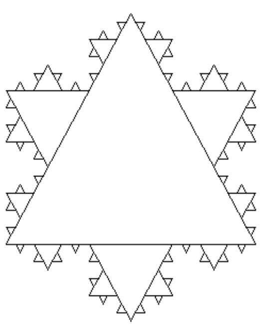
Fractals generated by iterated function systems. 

S is the decreasing limit of compact sets S_n.
It is defined by replacing one triangle (a_0, a_1, a_2) with three scaled copies.

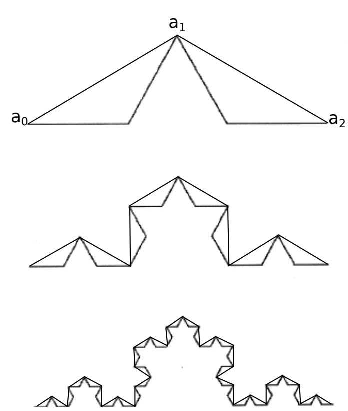
$$(a_0, a_1, a_2) \mapsto (a_0, \frac{a_0 + a_1}{2}, \frac{a_0 + a_2}{2})$$



$$M_1 := \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, M_2 := \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, M_3 := \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$



The fractal on the left hand side is generated by replacing one triangle with four scaled copies.

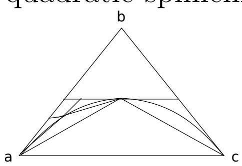


$$M_1 := \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, M_2 := \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix}, M_3 := \frac{1}{3}\begin{pmatrix} 0 & 3 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}, M_4 := \frac{1}{3}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

or simply

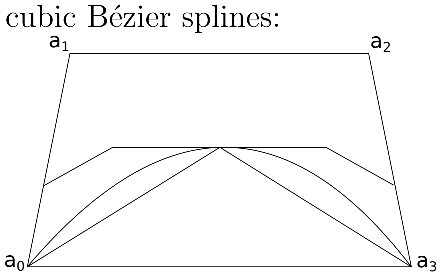
$$M := \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

quadratic splineinterpolation:



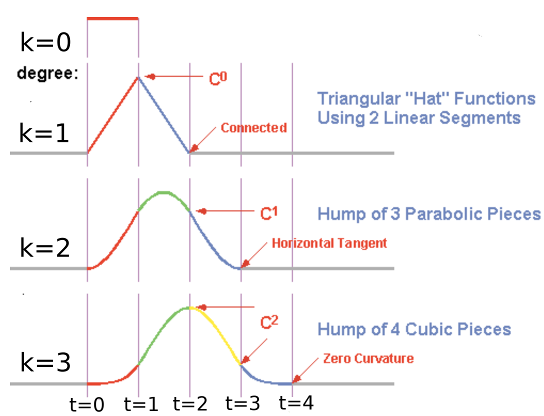
$$q_{\frac{1}{2}}(a,b,c) = \frac{1}{4}(a+2b+c), \qquad M_1 = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}, M_2 = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

cubic Bézier splines:



$$M_1 = \frac{1}{8}\begin{pmatrix} 8 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}, M_2 = \frac{1}{8}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

$S_k = A^k \circ D \leftrightarrow k^{\text{th}}$ - order B-spline



$\eta^0(t) = \chi_{[0,1]}$ (piecewise linear and C^0)

$$\eta^1(t) = \begin{cases} t & \text{if } 0 < t \leq 1 \\ -t & \text{if } 1 < t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(piecewise quadratic and C^1)

$\eta^k(t)$ is supported on $[0, k+1]$, is C^{k-1} and is polynomial of degree k on each $[i, i+1], i \in \mathbb{Z}$
Via convolution, inductively define: $\eta^{k+1} := \eta^k * \eta^0$,

$$\eta^{k+1}(t) = \int_{\mathbb{R}} \eta^k(t-n) \cdot \eta^0(n)dn = \int_0^1 \eta^k(t-n)dn$$

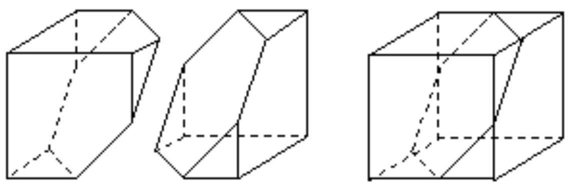
What is the k^{th} - order B-spline with controlpoints (a_n):

$$\gamma(t) = \sum_{n \in \mathbb{Z}} a_n \eta^k(t-n)$$

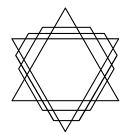
What is the B-spline in $\mathbb{R}^1$ with controlpoints $\ldots, 0, 0, 1, 0, 0 \ldots (a_n = \delta_{n0})$

$\eta^0(t) = \eta^0(2t) + \eta^0(2t-1)$
$\eta^1 = \eta^0 * \eta^0$
$\eta^k = \eta^0 * \eta^0 * \ldots * \eta^0$

$$\eta^k = \sum_{i=0}^{k+1} \frac{1}{2^k}\binom{k+1}{i}\eta^k(2t-i), \quad \eta^1(t) = \sum_{i=0}^{2} \frac{1}{2}\binom{2}{i}\eta^1(2t-i)$$



Slices through a cube:



$[0,1]^{k+1}$ unit cube in $\mathbb{R}^n$
$e = (1, 1, \ldots, 1) \in \mathbb{R}^n$
$H_t = \{x \in \mathbb{R}^{k+1}; < x, e >= \frac{t}{\sqrt{k+1}}\}$
k - area $(H_t \cap [0,1]^{k+1}) = \sqrt{k+1} \cdot \eta^k(t)$
$s_i^k := \frac{\binom{k+1}{i}}{2^k}$ ("subdivision mask")
$\eta^k(t) = \sum_{i=0}^{k+1} s_i^k \eta^k(2t-i)$
The B-spline $\sum a_n \eta^k(t-n) = \sum b_m \eta^k(2t-m)$ where $b_m := \sum s^k_{m-2n} a_n$
Indeed, we can check s^k_{m-2n} is the matrix for $S_k = A^k \circ D$

$k = 1$ (piecewise linear case): $S_1 = \frac{1}{2}\begin{pmatrix} & 0 & 2 & 0 & 0 & 0 & \\ \cdots & 0 & 1 & 1 & 0 & 0 & \cdots \\ & 0 & 0 & 2 & 0 & 0 & \\ \cdots & 0 & 0 & 1 & 1 & 0 & \cdots \\ & \vdots & & \vdots & & \vdots & \end{pmatrix}$

$k = 2$: $S_2 = \frac{1}{4}\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}$