Mathematical Visualization WS 12/13 lecture notes

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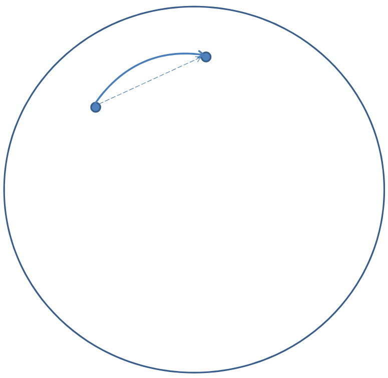
$$S^{n-1} := \{x \in \mathbb{E}^n \mid d(x,0) = 1\}$$

$d_{\mathbb{S}} = 2 \arcsin \frac{|p-q|}{2} = \arccos\langle p, q\rangle$ is the induced path metric.



Proposition 0.1:

$$O_n = \text{Isom}(S^{n-1})$$

Proof: If we are given an isometry on S^{n-1}, extand it linearly on rays through 0 to give a map $\varphi : \mathbb{E}^n \to \mathbb{E}^n$.

$$\begin{aligned} |\varphi(\lambda_p) - \varphi(\mu_q)|^2 &= |\lambda\varphi(p) - \mu\varphi(q)|^2 \\ &= \lambda^2 + \mu^2 - 2\lambda\mu\langle\varphi(p), \varphi(q)\rangle \\ &= \lambda^2 + \mu^2 - 2\lambda\mu\langle p, q\rangle \\ &= |\lambda p - \mu q|^2 \end{aligned}$$

□

This fixed point set of $\varphi \in E_n$ is

$$\text{Fix}(\varphi) := \{x \in \mathbb{E}^n | \varphi x = x\}.$$

Since φ is an affine map $\text{Fix}(\varphi)$ is an affine subspace of $\mathbb{E}^n$. We'll classify φ based on dim(Fix(p)) or on its codimension

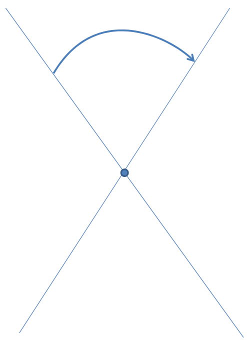
$$\text{codim}(\text{Fix}(\varphi) = n - \dim(\text{Fix}(\varphi).$$

- $\text{codim}\{p\} = n$
- $\text{codim}\{\emptyset\} = n + 1$ (convention)

codim Fix φ	dim	φ
0	n	id
1	$n-1$	$\mu_{\text{Fix}(\varphi)}$

- $\mu_{\text{Fix}(\varphi}$ = reflection across a hyperplane
- $\mu_{\{x_i=0\}}$ negates the i-th coordinate x_i
- $\mu_P = \varphi\mu_Q\varphi^{-1}$ for any $\varphi \in E_n$ maping Q to P.

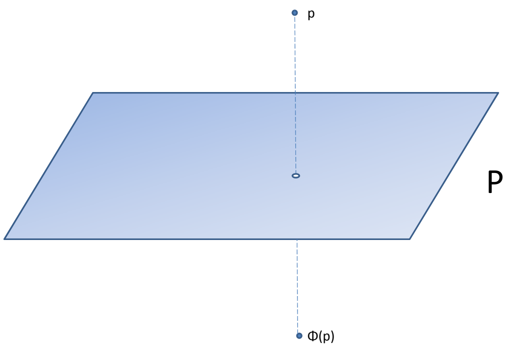
If P, Q are intersecting hyperplanes, $A = P \cap Q$ has codimension $= 2$. $\mu_P\mu_Q$ has fixed point set A and ever affine 2-plane $\perp A$ act as a rotation (by twice the angle between P, Q)



Definition 0.2: This is called a **simple rotation** with **axis A**.

If $P \neq Q$ are parallel hyperplanes $P = Q + v, v \perp P$. Then $\mu_P\mu_Q = \tau_{2v}$. For any $p, q \in \mathbb{E}^n \exists!$ reflection sending p to q (reflection across perpedicular bisector)
(Note: if $g^2 = e = h^2$ then $(gh)^{-1} = hg$)

Proposition 0.3: Suppose $\varphi \in E_n$ with $k = \text{codim}\,\text{Fix}(\varphi)$. Then φ can be written as a product of atmost k reflections.



Proof: By induction: $k = 0, 1$ done.
Suppose $A - \mathrm{Fix}(\varphi)$ has $\mathrm{codim}(A) = k > 0$. $\exists p \in \mathbb{E}^n : \varphi(p) \neq p$. If P is their $\perp$ projcetion (hyperplane), then $A \subset P$.

$$a \in A \Rightarrow d(a, p) = d(\varphi(a), \varphi(p)) = d(a, \varphi(p))$$

Consider $\psi := \mu_P \varphi$. This fixes all points in A and fixes p. $\mathrm{codim}(\mathrm{Fix}(\psi)) \leq k - 1$. By induktion ψ is the product of at most $k - 1$ reflections. So $\varphi = \mu_P \psi$ is the product of atmost k refletions. (we could add pairs of cancelling reflections to always use a product of $k - 1$ or k reflections (depending on orientation) or "n or nt"). □

Corollary 0.4:
(Euler) In 3D all rotations are simple - any $\varphi \in SO_3$ has a fixed point 0 so $\mathrm{codim}(\mathrm{Fix}(\varphi)) \leq 3$ but φ is orientation preserving, so its not product of 3 reflections.

$O_1 < E_1 \sim O_2 < E_2 \sim O_3 < E_3$ consider these and there discrete subgroups.
(Any orbit is a discrete topological subset of $\mathbb{E}^n$. Equivalent, these subgroups are discrete subspaces of the Lie groups O_n, E_n.)
The discret subgroups of O_n are exactly the finit subgroups.

- $O_1 = \{\pm 1\}$
- $SO_1 = \{e\}$
- elements of E_1 are:
  - reflections $\mu_x : p \mapsto 2x - p$
  - translations $\tau_x : p \mapsto x + p$ (product of 2 reflections)
- discrete subgroups G of E_1:
  - no translation - $G = SO_1$ or O_1
  - small translation τ_h then

    $$G > \langle \tau_h \rangle = \{\tau_h \mid h \in \mathbb{Z}\} \simeq \mathbb{Z}$$

    if $G < SE$ then $G = \langle \tau_h \rangle$ we proofed:

    $$G \cap SE = \langle \tau_h \rangle$$

    If $G \not< SE$ then $G = \langle \tau_h, \mu_a \rangle \simeq D_\infty$ infinit diheadral group.

These subgroups $\langle \tau_h \rangle, \langle \tau_h, \mu_a \rangle$ depends on h, a.
Changing a is conjugation by a translation.
Changing h is conjugation by a scale.
Up to conjugation in the group of similarities (affine transformation of $\mathbb{R}$) there are just 4 discrete subgroups: $E_1 : SO_1 = \{e\}, O_1 = \mathbb{Z}_2, \langle \tau_1 \rangle = \mathbb{Z}, \langle \tau_1, \mu_0 \rangle = D_\infty$

Now consider O_2. Again any element is the product of ≤ 2 reflections $\Rightarrow$ reflections, translations.
Thinking of $S^1 \subset \mathbb{C}$:

$$\mu_\theta : z \mapsto \frac{e^{2i\theta}}{z} = e^{2i\theta}\bar{z}$$
$$\rho_\theta : z \mapsto e^{i\theta} z$$

Finit subgroups of SO_2: $C_n = \langle \rho_{2\pi/n} \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ ($n = 1, 2, \dots$) for finit $G < O_2, G \cap SO_2 = C_n$ for some n.

If $G \not\leq SO_2$, then $G = \langle \rho_{2\pi/n}, \mu_\theta \rangle$. Up to conjugation in O_2 can take $\theta = 0$. This group has $2n$ elements and is a diheadral group $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$.

Elements of E_2:

- 0: id
- 1: reflections μ_e
- 2: translation τ_v, rotation $\rho_{a,\theta}$
- 3: glides