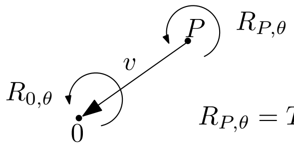
# Mathematical Visualization Lectures Nov. 12th - Nov. 15th

## Nov. 12th

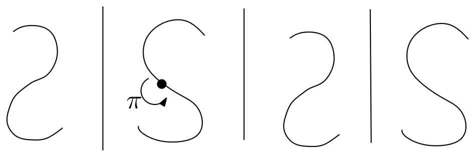
Recall from last week: $E(2)$ with $O(2)$ the orthogonal group and $\mathbb{R}(2)$ the translation group
$\phi : E(2) \to O(2)$ with $\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$ is a group homomorphism
It is: $\ker(\phi) = \mathbb{R}(2)$, i.e. $\mathbb{R}(2) \trianglelefteq E(2)$.




$R_{P,\theta} = T_v \circ R_{0,\theta} \circ T_v^{-1}$

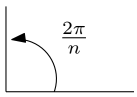
Remember that $h \mapsto ghg^{-1}$ is the conjugation by $g$. If you put in a translation it stays a translation. This is another way to see that the translation group is a normal subgroup.

*Discrete subgroup*: the identity element is *isolated*, i.e. it has a neighborhood with no other group element.

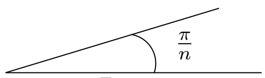



$\dim \ker \phi$ gives a classification of discrete subgroups
Let $\Gamma \leq E(2)$ be a discrete subgroup. Then $\phi(\Gamma) \subset O(2)$ and define $T := \ker \phi(\Gamma) \subset \mathbb{R}(2)$
$\dim T = 0, 1, 2$

0: point ($C_n$ or $D_n$ where $C_n$ is generated by a $\frac{2\pi}{n}$ rotation and $D_n$ by two intersection reflections of angle $\frac{\pi}{n}$)




$C_n$

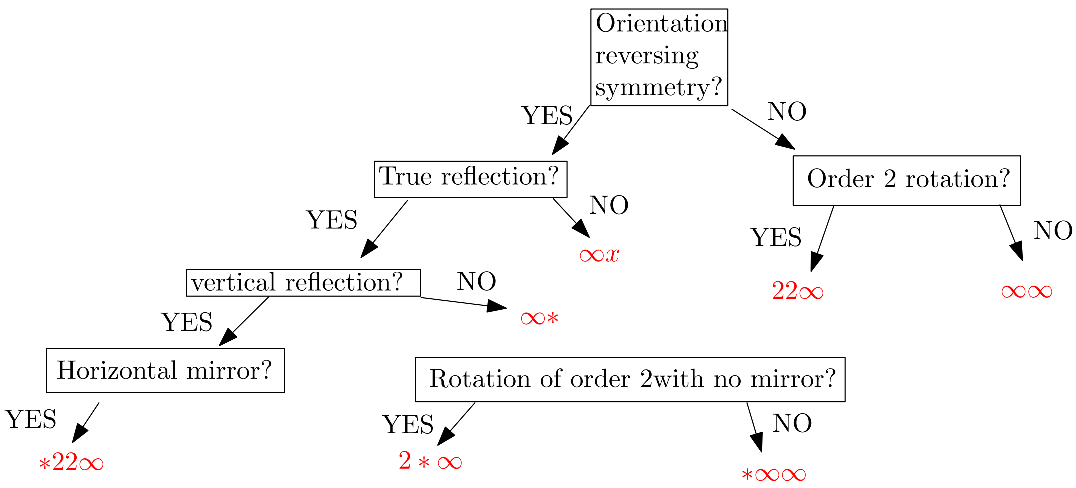



$D_n$

We have that $C_n \trianglelefteq D_n$ and $[D_n : C_n] = 2$

1: frieze groups (there are seven of them)
2: wallpaper groups (or in general: crystallographic groups) there are 17 of them

## Decision tree for patterns to recognize frieze groups

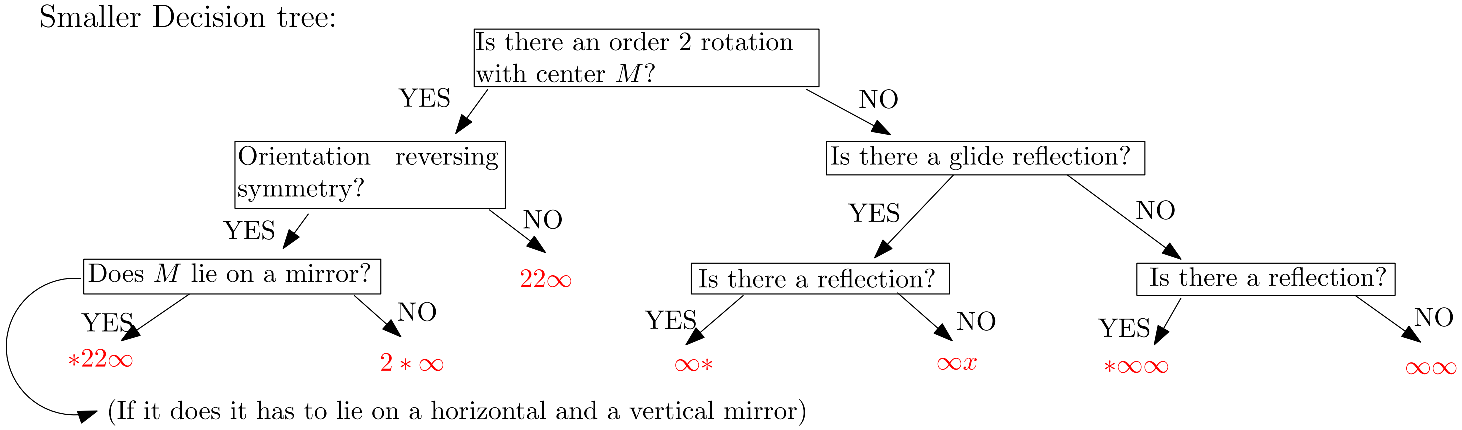



Question: Is there a cleaner set of questions? (there is, as we'll see tomorrow)
Wallpaper groups: order: 2, 3, 4, 6, $C_n$ or $D_n$
orientation preserving? YES $C_n$, id, NO $D_n$
Three groups with $D_1$ (just one reflection): $xx, **, *x$

| | | | |
|---|---|---|---|
| $C_1$ | 0 | $D_1$ | $**, xx, *x$ |
| $C_2$ | 2222 | $D_2$ | $*2222, 2*22, 22*, 22x$ |
| $C_3$ | 333 | $D_3$ | $*3333, 3*3$ |
| $C_4$ | 244 | $D_4$ | $4*2, *244$ |
| $C_6$ | 236 | $D_6$ | $*236$ |

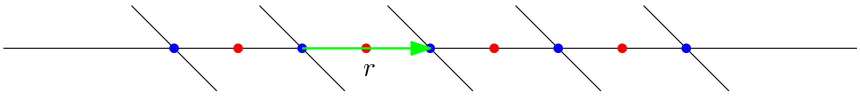
## Nov. 13th (Problem class)

Smaller Decision tree:




(If it does it has to lie on a horizontal and a vertical mirror)

$T \triangleleft G$ discrete group $\leq E(2)$
$g \in G$ : factorization of $g$
conjugacy classes $A$ of $T \triangleleft G$



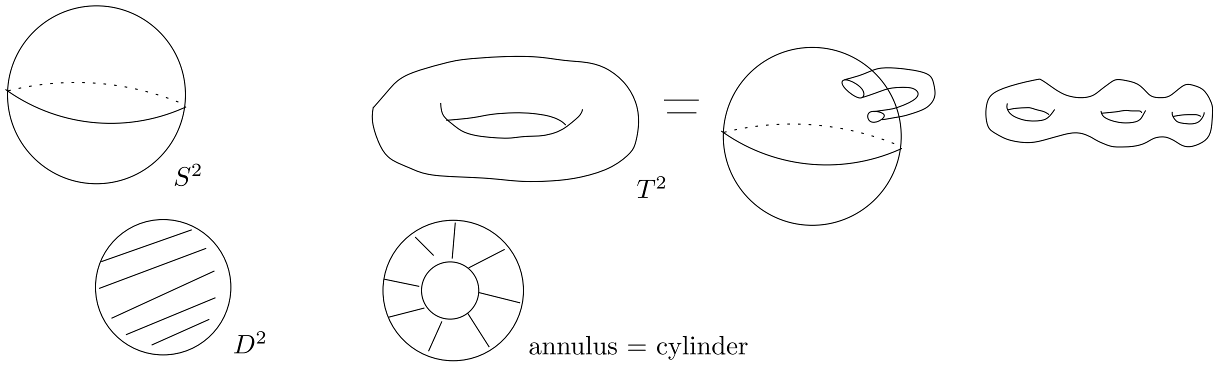

product of two rotations is translation

$T \triangleleft G, [G:T] = 2$
every element of $G$ can be written as: $T$ or $Tr$

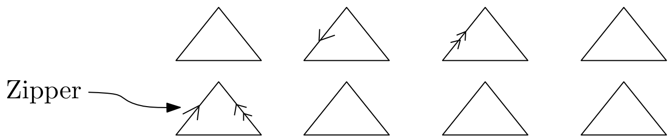
Then Charles explained what this means for Assignment 3.

## Nov. 15th

# Classification of compact surfaces

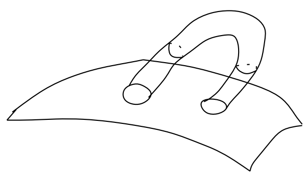



In higher dimensions, people distinguish topological, smooth and PL manifolds.
A theorem from the 1930's says there is no distinction in 2 or 3 dimensions.
For us, it means we can assume any compact surface is triangulated, i.e. it arises from glueing (finitely many) triangles together.
Zip-proof:

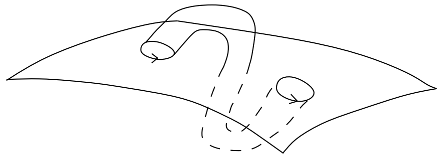



### Features in surfaces:

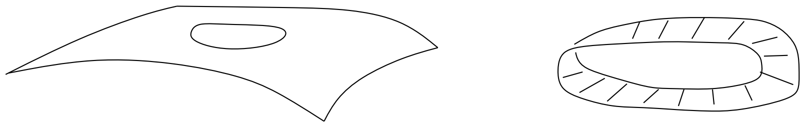
*puncture*: remove an open disk
*handle*: remove two open disks, sew in a cylinder



*cross-handle*: remove two disks, sew in cylinder this way



*cross-cap*: remove a disk, sew in Möbius band



sphere with a puncture: disk
sphere with two punctures: cylinder
sphere with a handle: torus
sphere with cross-handle: Klein-Bottle ($K^2$)
sphere with a cross-cap: $\mathbb{R}P^2$ (projective plane) sphere with two cross-caps: $K^2$

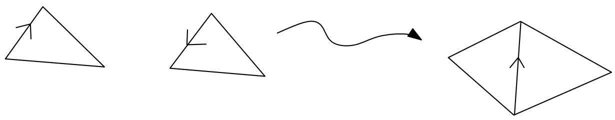
locally: adding two crosscaps = adding cross-handle

Temporary Definition:
An *ordinary* surface is a finite union of components, each being asphere with some number of punctures, handles, cross-handles, cross-caps added.

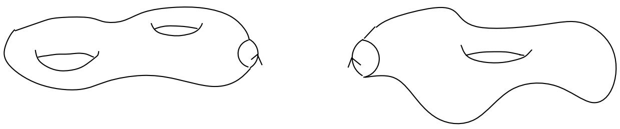
First goal: Every surface is ordinary.
Follows immediately from:

Lemma:
If we start with an ordinary surface and zip up one zipper, the result is ordinary.

Proof:

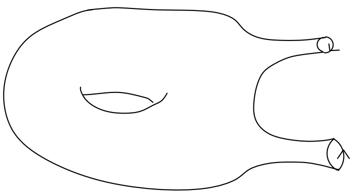


Suppose first the two sides of the zipper are full boundary circles.



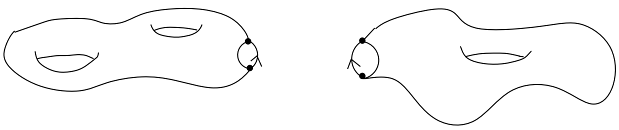
| | | | |
|---|---|---|---|
| $S^2$ with | $p$ | punctures | $p'$ |
| | $h$ | handles | $h'$ |
| | $k$ | cross-handles | $k'$ |
| | $c$ | cross-caps | $c'$ |

$p + p' - 2$ punctures
$h + h'$ handles
$k + k'$ cross-handles
$c + c'$ cross-caps

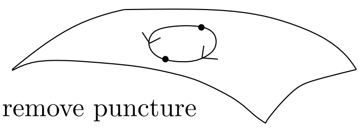


If the two sides of the zipper start on the same component, the zipping replaces two punctures by a handle or cross-handle.

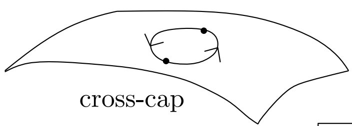
If the zippers do not go all the way around, the result is the same, except with a puncture left. This is also the case if one is a full loop and the other only a half.



What if the two sides of the zipper are the same boundary components?

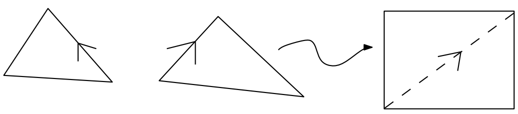


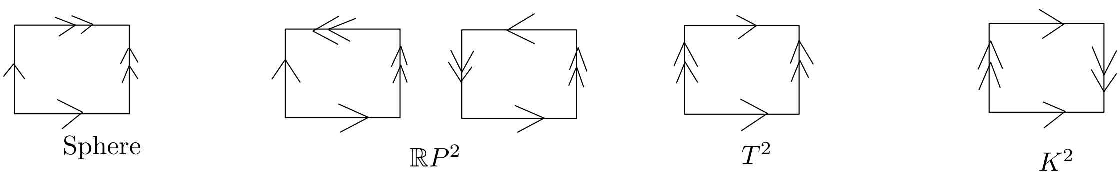
remove puncture

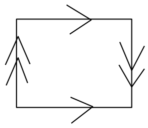


cross-cap

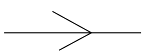
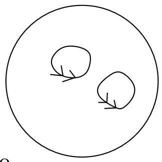
□

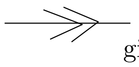


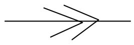


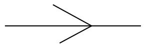


$K^2$

Zipping  first gives: 

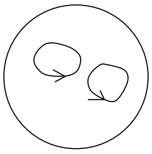
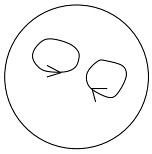
 gives cross-handle

Zipping  first gives Möbius band (i.e. $S^2$ with cross-cap and puncture)

Zipping  then adds a cross-cap

Result is $S^2$ with two cross-caps

On a non-orientable component (i.e. one with at least one cross-cap or cross-handle) there is no difference between adding a handle or a cross-handle.

### Final classification

Each connected compact surface is either a sphere with $g \geq 0$ handles and $k \geq 0$ punctures $\Sigma_{g,k}$ (if orientable) or if non-orientable is a sphere with $h \geq 1$ cross-caps and $k \geq 0$ punctures $N_{h,k}$.
$\Sigma_{0,2}$= cylinder, $N_{1,0} = \mathbb{R}P^2$, $N_{1,1}$= Möbius band, $N_{2,0} = K^2$.

## Euler characteristic

$$\chi = V - E + F$$

(after zipping)

$V$: # vertices, $E$: # edges, $F$: # triangles

$\chi(S^2) = 2$
$\chi(\Sigma_{g,k}) = 2 - 2g - k$
$\chi(N_{h,k}) = 2 - h - k$

The type of a connected compact surface can be determined from:
$k =$# boundary components, $\chi$, orientable?
If $\chi + k$ is odd, then the surface is non-orientable.